Completing The Square

Expand the following brackets.
Do you notice any relationship between the original expression and coefficient of x in the expanded expression in each case?

$$(x+1)^2 =$$
$$(x+3)^2 =$$
$$(x-4)^2 =$$
$$(x+a)^2 =$$

Therefore…

Therefore, it seems as if we can **halve the coefficient of $\boldsymbol{x}$** to get the missing number in $(x + \_)^2$

Put the following in the form $(x + a)^2 + b$

$x^2 + 10x \quad \rightarrow$

$x^2 + 2x \quad \rightarrow$

$x^2 + x \quad \rightarrow$

$x^2 - 12x \quad \rightarrow$

✎ Putting a quadratic expression in the form $(x+a)^2+b$ is known as '**completing the square**'.

Put the following in the form $(x+a)^2+b$

$x^2+8x+3 \quad \rightarrow$

$x^2-x+1 \quad \rightarrow$

Exercise

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Put the following in the form $(x + a)^2 + b$

a $x^2 + 7x + 2 \ =$

b $x^2 - \frac{1}{3}x + \frac{1}{4} \ =$

What if the coefficient of x^2 is not 1?

Put $2x^2 + 8x + 7$ in the form $a(x+b)^2 + c$

Put $3x^2 - 6x + 11$ in the form $a(x+b)^2 + c$

Put $3 - 12x - x^2$ in the form $a(x+b)^2 + c$

Put $\frac{1}{10}x^2 + 4x - 3$ in the form $a(x+b)^2 + c$

Why complete the square?

There are 3 major applications of completing the square, two of which we will look at:

1 :: Solving Equations	"(a) Write $x^2 + 4x - 7$ in the form $(x + a)^2 + b$. (b) Hence determine the exact solutions of: $x^2 + 4x - 7 = 0$
2 :: Finding the Turning Point of a Parabola	"Determine the turning point of the line with equation $y = x^2 - 4x + 5$"
3 :: (Further Maths A Level) Integrating reciprocals of quadratics	"Determine $\int \frac{1}{x^2-4x+5} dx$"

Application 1 :: Solving by Completing the Square

a) Write $x^2 + 4x - 7$ in the form $(x + a)^2 + b$
b) Hence, determine the exact solutions of $x^2 + 4x - 7 = 0$

a) Write $2x^2 + 8x - 1$ in the form $a(x + b)^2 + c$

b) Hence, determine the exact solutions of $2x^2 + 8x + 1 = 0$

Application 2 :: Minimum/Maximum Values

Completing the square also allows us to find the minimum or maximum value of a quadratic.

For the quadratic expression $x^2 - 4x + 11$,
(a) Determine its minimum value.
(b) Determine the value of x for which this minimum occurs.

Let's experiment with different values of x to see what gives us the smallest value:

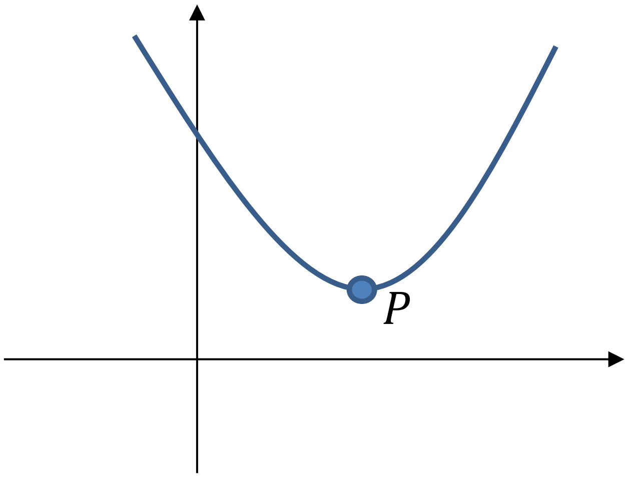
$x = 0 \quad \rightarrow$
$x = 1 \quad \rightarrow$
$x = 2 \quad \rightarrow$
$x = 3 \quad \rightarrow$

So the minimum value of $x^2 - 4x + 11$ appears to be which occurs when $x =$
But why is this?

For the quadratic expression $x^2 + 6x + 5$,
(a) Determine its minimum value.
(b) Determine the value of x for which this minimum occurs.

✎ The minimum value of $(x + a)^2 + b$ is b, which occurs when $x = -a$.

For the quadratic expression $-x^2 + 8x + 3$,
(a) Determine its **maximum** value.
(b) Determine the value of x for which this maximum occurs.



A curve with equation $y = x^2 - 8x + 17$ has a turning point at P. Determine the coordinates of P.

		Completed Square	Turning Point
$y = x^2 + 2x + 8$	$=$		
$y = x^2 - 6x + 3$	$=$		
$y = x^2 + 10x + 4$	$=$		
$y = 2x^2 + 8x + 1$	$=$		

Exercise

1. Find the turning point of the curves with the following equations:

$$y = 8 - 4x - x^2$$

2. Find the maximum value of the following expressions:

$$-3x^2 + ax + c$$

"Complete the square using $x^2 + 4x - 12$."

Student Answer: "$(x+6)(x-2)$"

What's wrong: The student factorised rather than completed the square.

- We **factorise** when we want to find the **<u>roots</u>** of a quadratic.
- Meanwhile, we **complete the square** when we want to find the **<u>turning point</u>** of a quadratic.

$x^2 - 6x + 11 \quad \rightarrow \quad (x-3)^2 + 9 + 11 = \cdots$

What's wrong: The 9 should have been subtracted (and it should always be a subtraction regardless of whether the coefficient of x is positive or negative.

"Write $2x^2 + 8x + 11$" in the form $a(x+b)^2 + c$"

Student Answer:

$$= 2(x^2 + 4x) + 11$$
$$= 2\left((x+2)^2 - 4\right) + 11$$
$$= 2(x+2)^2 + 7$$

What's wrong: In the last line of working, they did $-4 + 11 \quad \rightarrow \quad +7$
They forgot to multiply the -4 by 2.

Proof of the Quadratic Formula

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By completing the square, show that the solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$