P1 Chapter 1: Algebraic Expressions

Chapter Overview

1: Basic Index Laws

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{mn}$$

2: Expand brackets

Expand and simplify

$$(x-3)^2(x+1)$$

3: Factorise quadratics/cubics

Factorise fully:

$$x^3 - 16x$$

4: Fractional/Negative Powers

Evaluate:

$$\left(\frac{27}{8}\right)^{-\frac{2}{3}}$$

5: Surds

Simplify

$$\sqrt{12} \times 2 \quad = 6\sqrt{3}$$

6: Rationalising denominators

Rationalise the denominator of

$$\frac{1}{3+\sqrt{2}}$$

1: Index Laws

base → 3^5 ← exponent or power or index (plural: indices)

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

Simplify $(a^3)^2 \times 2a^2$

Simplify $(4x^3y)^3$

Simplify $2x^2(3+5x) - x(4-x^2)$

Simplify $\frac{x^3-2x}{3x^2}$

Worked Examples

1 Simplify $\left(\frac{2a^5}{a^2}\right)^2 \times 3a$

2 Simplify $\frac{2x+x^5}{4x^3}$

3 Expand and simplify

$$2x(3-x^2)-4x^3(3-x)$$

4 Simplify $2^x \times 3^x$

Non-Calculator Questions

1 Which of the following numbers is largest?

- $\left(\left(2^3\right)^2\right)^3$
- $\left(2^3\right)^{\left(2^3\right)}$
- $2^{\left(\left(3^2\right)^3\right)}$
- $2^{\left(3^{\left(2^3\right)}\right)}$

2 Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers.

Then N will definitely be a square number whenever:

- k is even;
- $k + n$ is odd;
- k is odd but $m + n$ is even;
- $k + n$ is even.

Multiply each term in the first bracket by each term in the second.

$$(x - y)(x + y - 1)$$

$$(x + 1)(x + 2)(x + 3)$$

Worked Examples

1 Expand and simplify

$$(x+5)(x-2)(x+1)$$

2 Expand and simplify:

$$2(x-3)(x-4)$$

3 Expand and simplify:

$$(2x-1)^3$$

Challenge Problems

Non-Calculator Questions

1

Of the following three alleged algebraic identities, at least one is wrong.

(i) $yz(z-y)+zx(x-z)+xy(y-x)$
$=(z-y)(x-z)(y-x)$

(ii) $yz(z-y)+zx(x-z)+xy(y-x)$
$=(z-y)(z-x)(y-x)$

(iii) $yz(x+y)+zx(z+x)+xy(y+x)$
$=(z+y)(z+x)(y+x)$

Which of the following statements are correct? Tick all that apply.

- (i)
- (ii)
- (iii)

2

If x and n are integers then

$$(1-x)^{n}(2-x)^{2n}(3-x)^{3n}(4-x)^{4n}(5-x)^{5n}$$

is:

- negative when $n>5$ and $x<5$
- negative when n is odd and $x>5$
- negative when n is a multiple of 3 and $x>5$
- negative when n is even and $x<5$

Informally, factorising is the opposite of expanding brackets.
More formally, a factorised expression is one which is expressed as **a product of expressions**.

$x(x+1)(x+2)$ ➡ Factorised as it is the product of 3 linear factors, $x, x+1$ and $x+2$.

$x(x+1)+(x-1)(x+1)$ ➡ Not factorised because the outer-most operation is a sum, not a product.

Basic Examples:

$$x^3 + x^2 =$$
$$4x - 8xy =$$

Recap:

$\oplus$ $\otimes$

$$x^2 - 5x - 14 =$$

We find two numbers which multiply to give the coefficient of x and multiply to give the constant term.

But what if the coefficient of x^2 is not 1?

$$2x^2 + 5x - 12 =$$

$$2x^2 + 5x - 12$$

$\oplus$

$\otimes$

$=$

$=$

$=$

STEP 1: Find two numbers which add to give the middle number and multiply to give the first times last.

STEP 2: Split the middle term.

STEP 3: Factorise first half and second half ensuring bracket is duplicated.

STEP 4: Factorise out bracket.

Other Factorisations

Difference of two squares:

$$4x^2 - 9 =$$

Using multiple factorisations:

$$x^3 - x$$

$$=$$

$$=$$

$$x^3 + 3x^2 + 2x$$

$$=$$

$$=$$

Worked Examples

1

Factorise completely:

$$6x^2 + x - 2$$

2

Factorise completely:

$$x^3 - 7x^2 + 12x$$

Worked Examples

3 Factorise completely:

$$x^4 - 1$$

4 Factorise completely:

$$x^3 - 1$$

4 : Negative and Fractional Indices

$$a^0 = 1$$
$$a^{\frac{1}{m}} = \sqrt[m]{a}$$
$$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$$
$$a^{-m} = \frac{1}{a^m}$$

Note: $\sqrt{9}$ only means the positive square root of 9, i.e. 3 not -3.
Otherwise, what would be the point of the $\pm$ in the quadratic formula before the $\sqrt{b^2 - 4ac}$?

Prove that $x^{\frac{1}{2}} = \sqrt{x}$

Evaluate $27^{-\frac{1}{3}}$

Evaluate $32^{\frac{2}{5}}$

4 : Negative and Fractional Indices

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Note: $\sqrt{9}$ only means the <u>positive</u> square root of 9, i.e. 3 not -3.
Otherwise, what would be the point of the $\pm$ in the quadratic formula before the $\sqrt{b^2 - 4ac}$?

Simplify $\left(\frac{1}{9}x^6y\right)^{\frac{1}{2}}$

Evaluate $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

If $b = \frac{1}{9}a^2$, determine $3b^{-2}$ in the form ka^n where k, n are constants.

Non-Calculator Questions

Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

is an integer if

- $r + s \leq 0$
- $s \leq 0$
- $r \leq 0$
- $r \geq s$

5 : Surds

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Recap:

A surd is a root of a number that does not simplify to a rational number.

Laws:

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

A *rational* number is any which can be expressed as $\frac{a}{b}$ where $a, b \neq 0$ are integers. $\frac{2}{3}$ and $\frac{4}{1} = 4$ are rational numbers, but π and $\sqrt{2}$ are not.

$\sqrt{3} \times 2 =$

$3\sqrt{5} \times 2\sqrt{5} =$

$\sqrt{8} = \sqrt{4}\sqrt{2} =$

$\sqrt{12} + \sqrt{27} =$

$(\sqrt{8} + 1)(\sqrt{2} - 3) =$

$=$

$=$

Challenge Problems

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Non-Calculator Questions

1 [SMC 2014 Q24] Which of the following is smallest?

- $10 - 3\sqrt{11}$
- $8 - 3\sqrt{7}$
- $5 - 2\sqrt{6}$
- $9 - 4\sqrt{5}$
- $7 - 4\sqrt{3}$

2 [SMC 2012 Q21] Which of the following numbers does *not* have a square root in the form $x + y\sqrt{2}$, where x and y are positive integers?

- $17 + 12\sqrt{2}$
- $22 + 12\sqrt{2}$
- $38 + 12\sqrt{2}$
- $54 + 12\sqrt{2}$
- $73 + 12\sqrt{2}$

Here's a surd. What could we multiply it by such that it's no longer an irrational number?

$$\sqrt{5} \times \square = \square$$

$$\frac{1}{\sqrt{2}} \times \square = \square$$

Examples

$$\frac{3}{\sqrt{2}} =$$

$$\frac{12}{\sqrt{3}} =$$

$$\frac{6}{\sqrt{3}} =$$

$$\frac{2}{\sqrt{6}} =$$

$$\frac{7}{\sqrt{7}} =$$

$$\frac{4\sqrt{2}}{\sqrt{8}} =$$

$$\frac{15}{\sqrt{5}} + \sqrt{5} =$$

More Complex Denominators

You've seen 'rationalising a denominator', the idea being that we don't like to divide things by an irrational number.

But what do we multiply the top and bottom by if we have a more complicated denominator?

$$\frac{1}{\sqrt{2}+1} \times \square = \square$$

More Examples

$$\frac{3}{\sqrt{6}-2} \times \square = \square$$

$$\frac{4}{\sqrt{3}+1} \times \square = \square = \square$$

$$\frac{3\sqrt{2}+4}{5\sqrt{2}-7} \times \square = \square$$

$$=$$

Worked Examples

Rationalise the denominator and simplify

$$\frac{4}{\sqrt{5}-2}$$

Rationalise the denominator and simplify

$$\frac{2\sqrt{3}-1}{3\sqrt{3}+1}$$

Solve $y\left(\sqrt{3}-1\right) = 8$

Give your answer in the form $a + b\sqrt{3}$ where a and b are integers.

Practice Test Questions

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1 Expand and simplify:

$$(\sqrt{5}+3)(\sqrt{5}-2)(\sqrt{5}+1)$$

2 Rationalise the denominator, giving your answer in the form $a+b\sqrt{3}$.

$$\frac{3\sqrt{3}+7}{3\sqrt{3}-5}$$

Practice Test Questions

TOPMaThs A* Level

3 Solve $y(1+\sqrt{2})-\sqrt{2}=3$

4 Simplify:

$$\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}+\sqrt{a}}$$

Challenge Problems

Non-Calculator Question

Solve $\dfrac{\sqrt[4]{9}}{\sqrt[5]{27}} = \sqrt[x]{3}$

1

Answer this question without a calculator, showing all your working and giving your answers in their simplest form.

(i) Solve the equation

$$4^{2x+1} = 8^{4x}$$

(3)

(ii) (a) Express

$$3\sqrt{18} - \sqrt{32}$$

in the form $k\sqrt{2}$, where k is an integer.

(2)

(b) Hence, or otherwise, solve

$$3\sqrt{18} - \sqrt{32} = \sqrt{n}$$

(2)

2

Answer this question without the use of a calculator and show your method clearly.

(i) Show that

$$\sqrt{45} - \frac{20}{\sqrt{5}} + \sqrt{6}\sqrt{30} = 5\sqrt{5}$$

(2)

(ii) Show that

$$\frac{17\sqrt{2}}{\sqrt{2} + 6} = 3\sqrt{2} - 1$$

(3)

3

Simplify the following expressions fully.

(a) $\left(\frac{1}{9}x^4\right)^{0.5}$

(1)

(b) $\left(\frac{x}{\sqrt{2}}\right)^{-2}$

(1)

(c) $x\sqrt{3} \div \sqrt{\frac{48}{x^4}}$

(2)