P1 Chapter 2 : Quadratics

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Chapter Overview

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1: Solving quadratic equations

Solve
$(x+1)^2 - 3(x+1) + 2 = 0$

2: Completing the square

Write $2x^2 + 8x - 5$ in the form $a(x+b)^2 + c$

3: Quadratics as functions

If $f(x) = x^2 + 2x$, find the roots of $f(x)$.

4: Quadratic Graphs

Sketch $y = x^2 + 4x - 5$, indicating the coordinate of the turning point and any intercepts with the axes.

5: The Discriminant

Find the range of values of k for which $x^2 + 4x + k = 0$ has two distinct real solutions.

6: Modelling with Quadratics

1 : Solving Quadratic Equations

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$$x^2 + 5x = 6$$

By factorisation

Using the quadratic formula.

1 : Solving Quadratic Equations

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Solving without factorising

If the subject only appears once however, it might be easier not to expand out/factorise:

$$(x-1)^2 = 5$$

Quadratics 'in disguise'

Solve $x - 6\sqrt{x} + 8 = 0$

substitution

hardcore

Worked Examples

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1 Solve $(x+3)^2 = x+5$ using factorisation.

2 Solve $(2x+1)^2 = 5$

3 Solve $\sqrt{x+3} = x-3$

4 Solve $2x + \sqrt{x} - 1 = 0$

Non-Calculator Questions

(i) Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

Challenge Problems

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Non-Calculator Questions

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2} - 22 = 0$;

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 = 0$.

Spot the quadratic equation

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1) $x^4 - 3x^2 - 4 = 0$

2) $x - 4\sqrt{x} + 3 = 0$

3) $x^{\frac{1}{4}} - 2x^{\frac{1}{2}} + 1 = 0$

4) $3\sin^2 x + 2\sin x - 1 = 0$

Spot the quadratic equation

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5) $x^4 = 10x^2 - 9$

6) $\frac{4}{x} + \frac{3}{x^2} + 1 = 0$

7) $\cos^2 x - 2\cos x = 8$

8) $3\tan x = 3 - \tan^2 x$

2 : Completing the Square

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"Completing the square" means putting a quadratic in the form $(x + a)^2 + b$ or $a(x + b)^2 + c$

a. Solving Quadratics

If we have a completed square:

$$(x + 4)^2 - 7 = 0$$

b. Sketching Quadratics

if $y = (x + a)^2 + b$, then the minimum point is $(-a, b)$

c. In integration

In Further Maths, completing the square allows us to 'integrate' expressions like:

$$\int \frac{1}{x^2 - 4x + 5} dx$$

Completing the Square Recap

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Expand:

$(x+9)^2 =$

$(x-5)^2 =$

Note:

$$x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$$

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

Therefore if we had x^2+12x, how could we write it in the form $(x+a)^2+b$?

$x^2+12x =$

Further Examples:

$x^2+8x =$

$x^2-2x =$

$x^2-6x+7 =$

$=$

Completing the Square

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Express $2x^2 + 12x + 7$ in the form $a(x + b)^2 + c$

$=$

$=$

$=$

$=$

Express $5 - 3x^2 + 6x$ in the form $a - b(x + c)^2$

$=$

$=$

$=$

$=$

$=$

Worked Examples

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Express $3x^2 - 18x + 4$ in the form $a(x + b)^2 + c$

$=$

$=$

$=$

$=$

Express $20x - 5x^2 + 3$ in the form $a - b(x + c)^2$

$=$

$=$

$=$

$=$

$=$

Solving by Completing the Square

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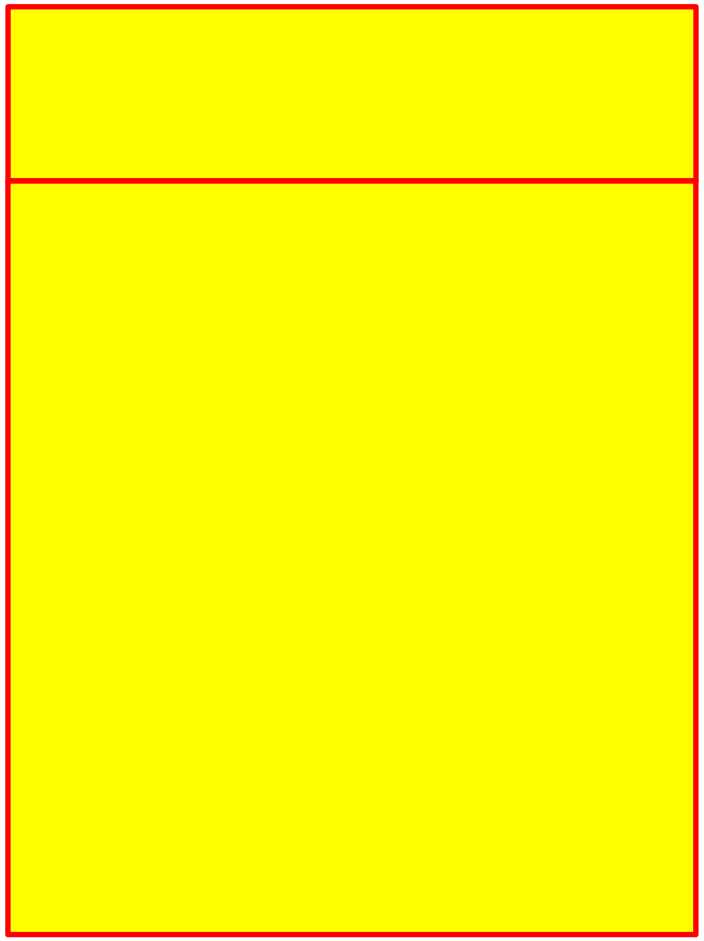
Solve the equation:

$$3x^2 - 18x + 4 = 0$$



Proving the Quadratic Formula

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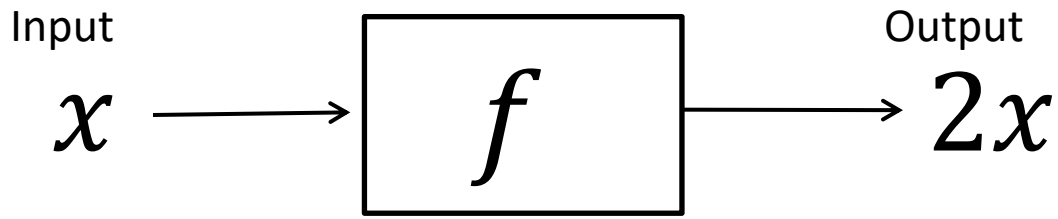
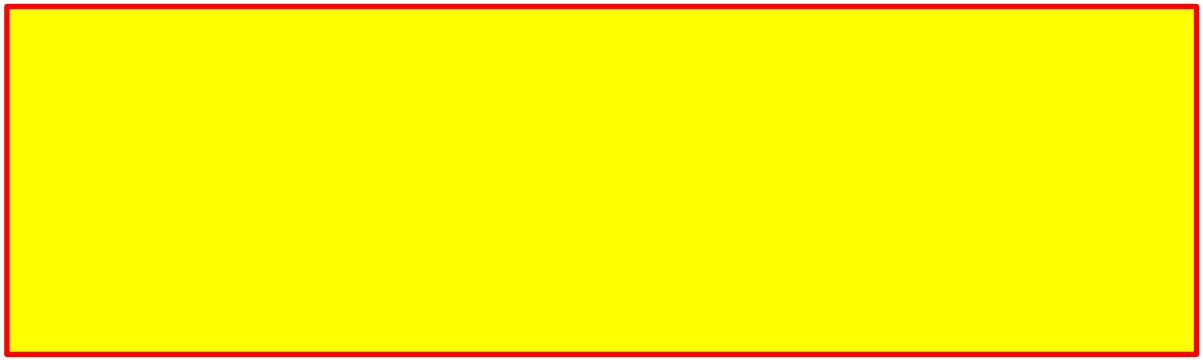
If $ax^2 + bx + c = 0$, prove that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

A function is something which **provides a rule on how to map inputs to outputs**.

We saw at GCSE that functions were a formal way of describing a 'number machine':





3 : Functions

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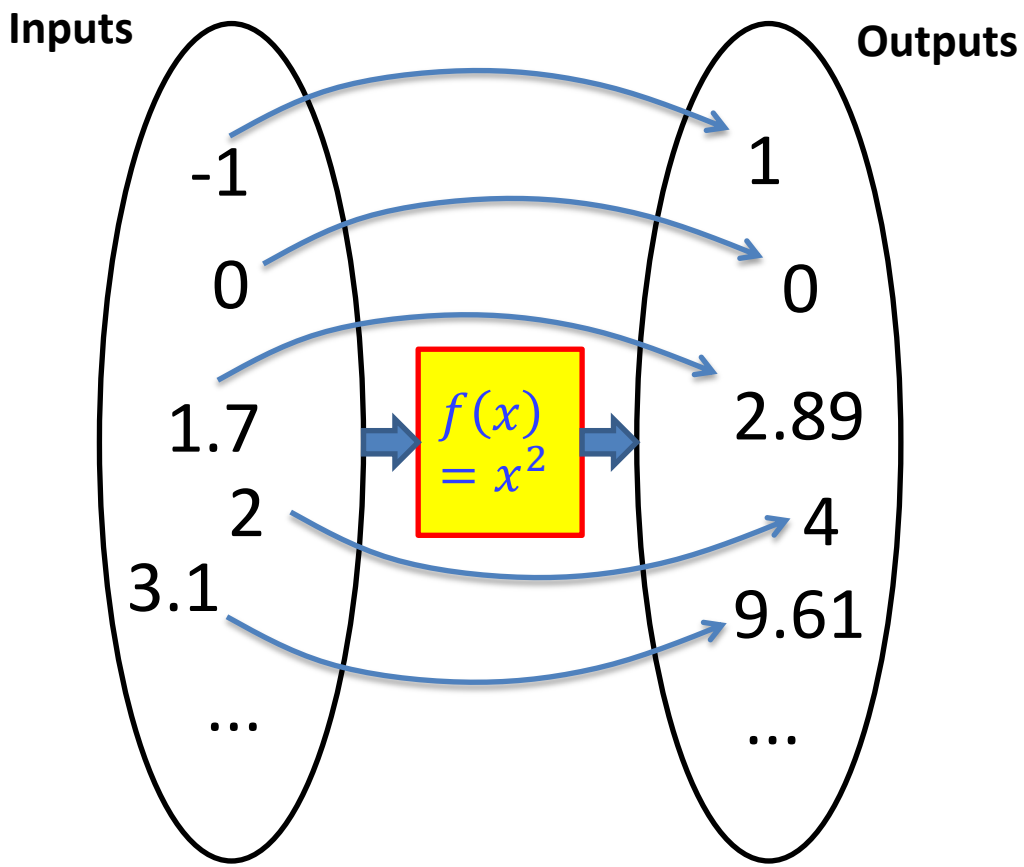

The domain of a function could potentially be **any** real number. If so, we'd write:

We might be interested in what inputs x give an output of 0. These are known as the of the function.

✎ The **domain** of a function is the set of possible inputs.

✎ The **range** of a function is the set of possible outputs.

✎ The **roots/zeroes** of a function are the values of x for which $f(x) = 0$.

Worked Examples

If $f(x) = x^2 - 3x$ and $g(x) = x + 5, \quad x \in \mathbb{R}$

a) Find $f(-4)$

b) Find the values of x for which $f(x) = g(x)$

c) Find the roots of $f(x)$.

d) Find the roots of $g(x)$.

a

b

c

d

Examples

Determine the minimum value of the function $f(x) = x^2 - 6x + 2$, and state the value of x for which this minimum occurs.

$f(x)$	Completed square	Min/max value of $f(x)$	x for which this min/max occurs
$x^2 + 4x + 9$			
$x^2 - 10x + 21$			
$10 - x^2$			
$8 - x^2 + 6x$			

Worked Examples

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1 Find the minimum value of $f(x) = 2x^2 + 12x - 5$ and state the value of x for which this occurs.

2 Find the roots of the function $f(x) = 2x^2 + 3x + 1$

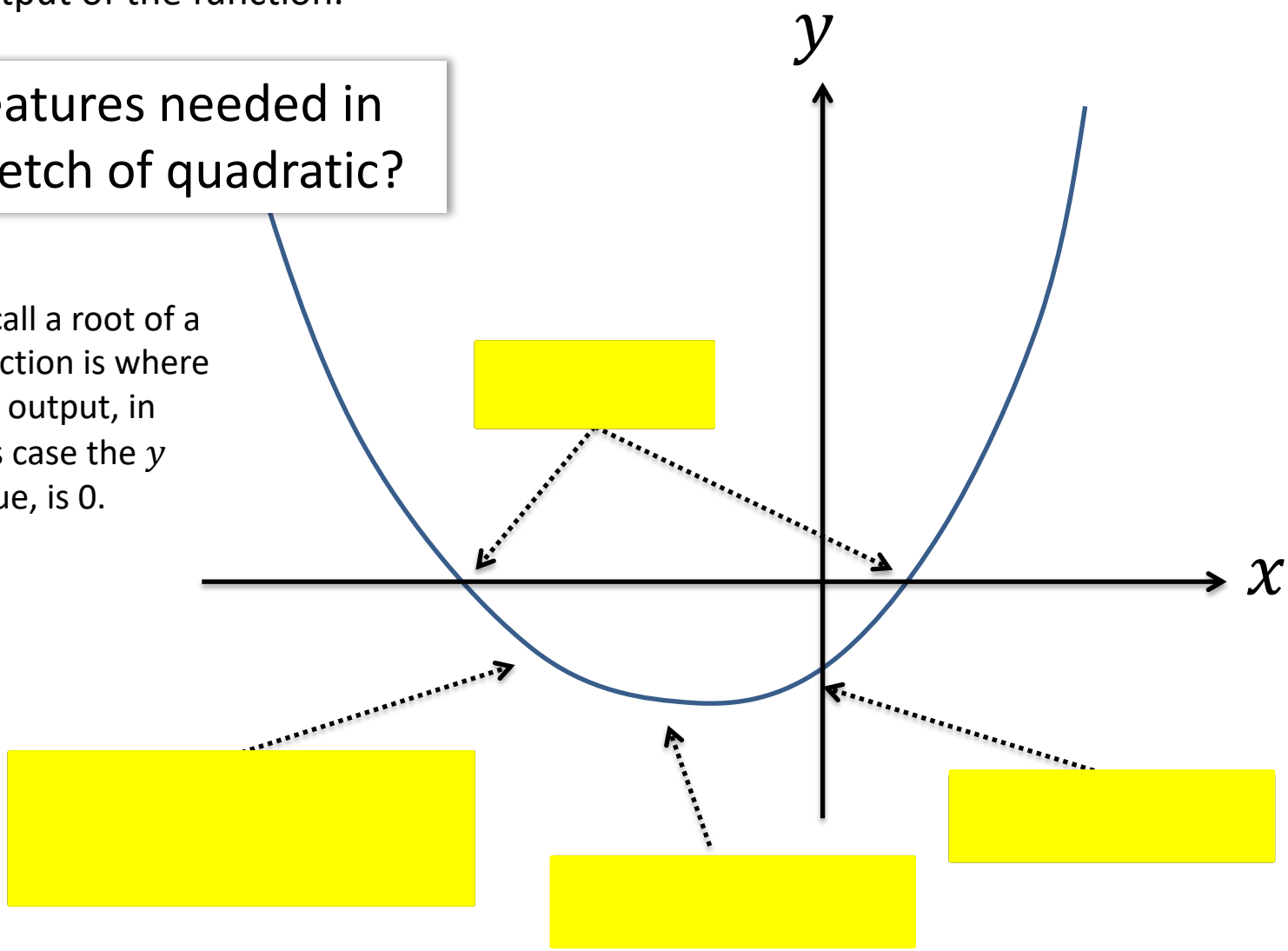
3 Find the roots of the function $f(x) = x^4 - x^2 - 6$

4 : Quadratic Graphs

Recall that x refers to the input of a function, and the expression $f(x)$ refers to the output. For graph sketches, we often write $y = f(x)$, i.e. we set the y values to be the output of the function.

Features needed in sketch of quadratic?

Recall a root of a function is where the output, in this case the y value, is 0.



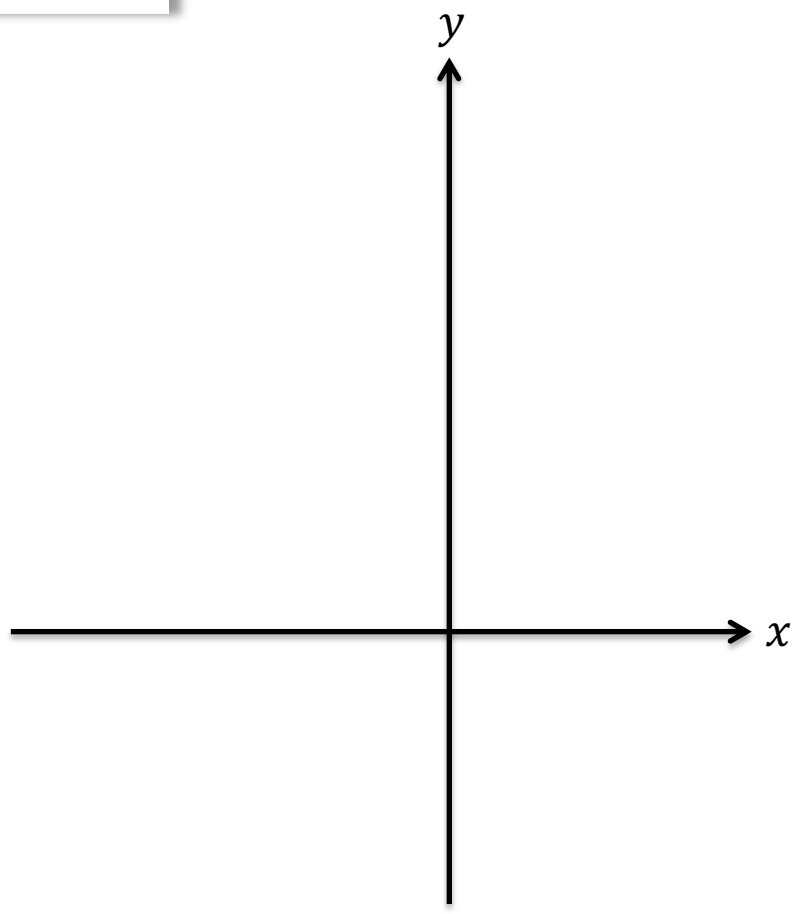
Example

Sketch the graph of $y = x^2 + 3x - 4$ and find the coordinates of the turning point.

Roots:

y-intercept:

Turning point:



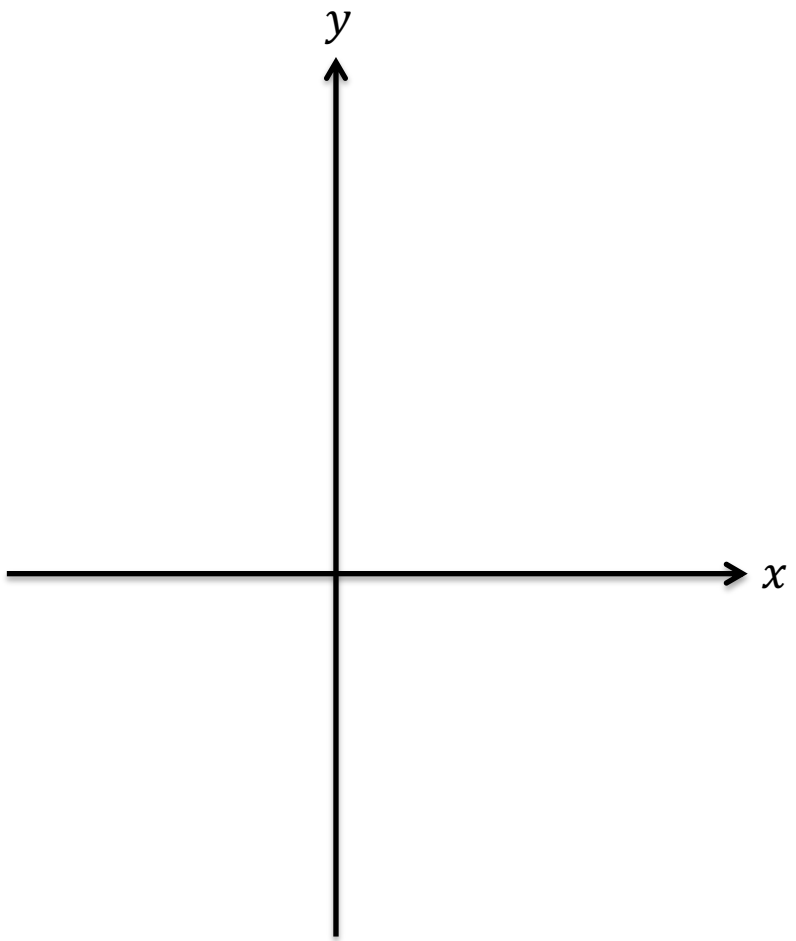
Example

Sketch the graph of $y = 4x - 2x^2 - 3$ and find the coordinates of the turning point. Write down the equation of the line of symmetry.

Roots:

y-intercept:

Turning point:



Line of reflection:

Practice Test Questions

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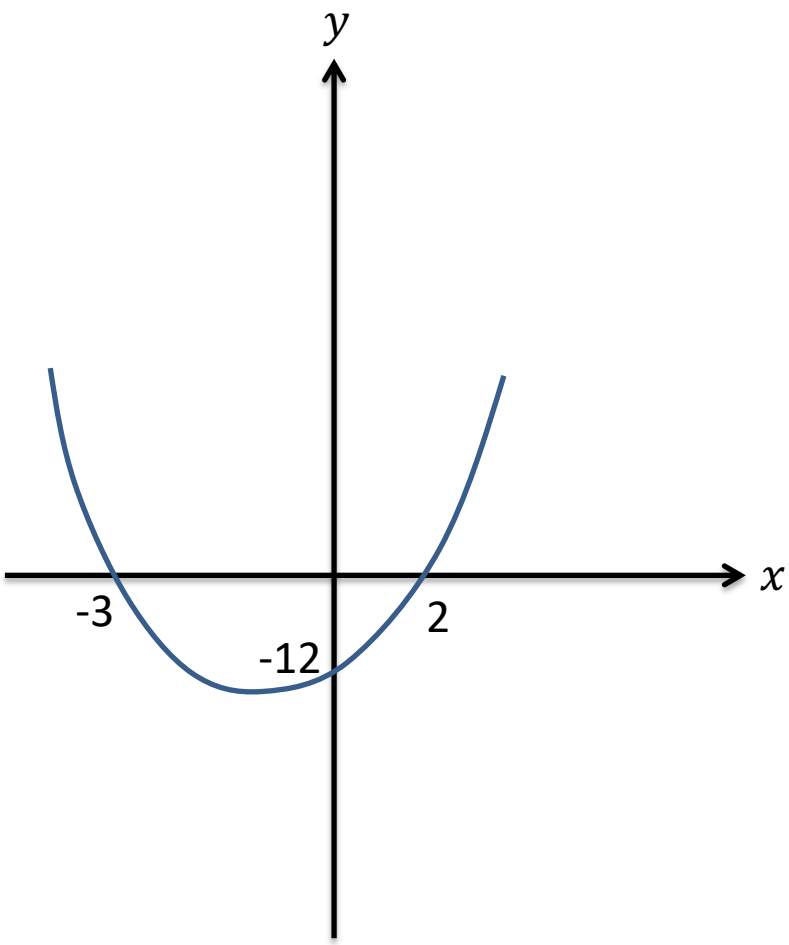
Sketch the following, indicating any intercepts with the axis, the turning point and the equation of the line of symmetry.

a $y = x^2 + 4$

b $y = x^2 - 7x + 10$

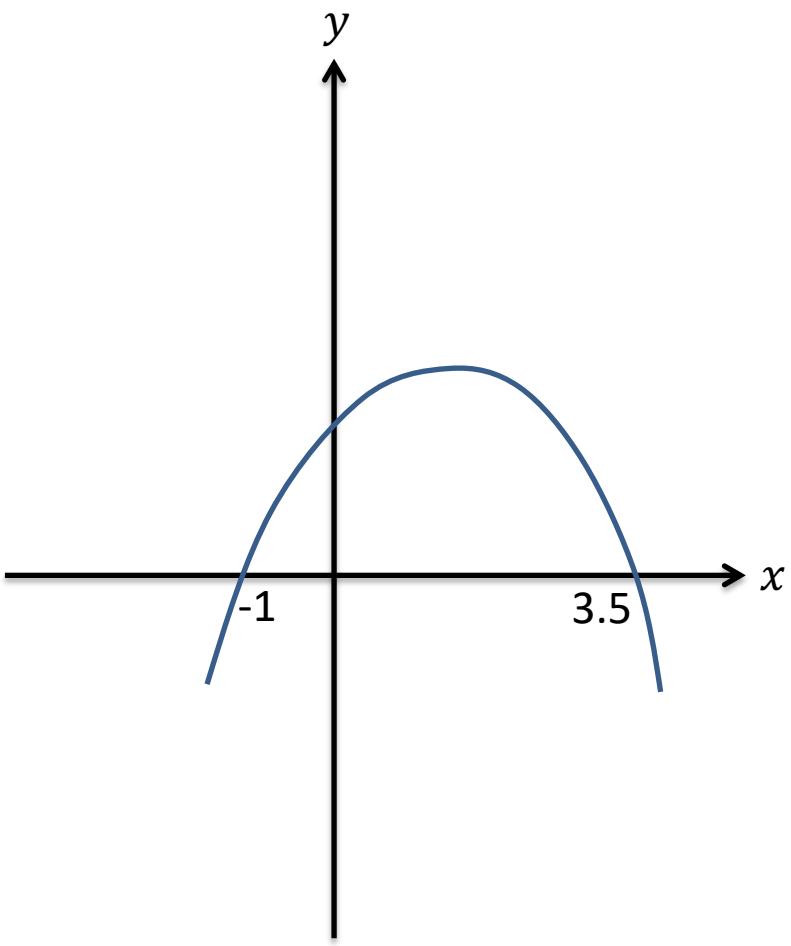
c $y = 5x + 3 - 2x^2$

Determining the Equation using a Graph



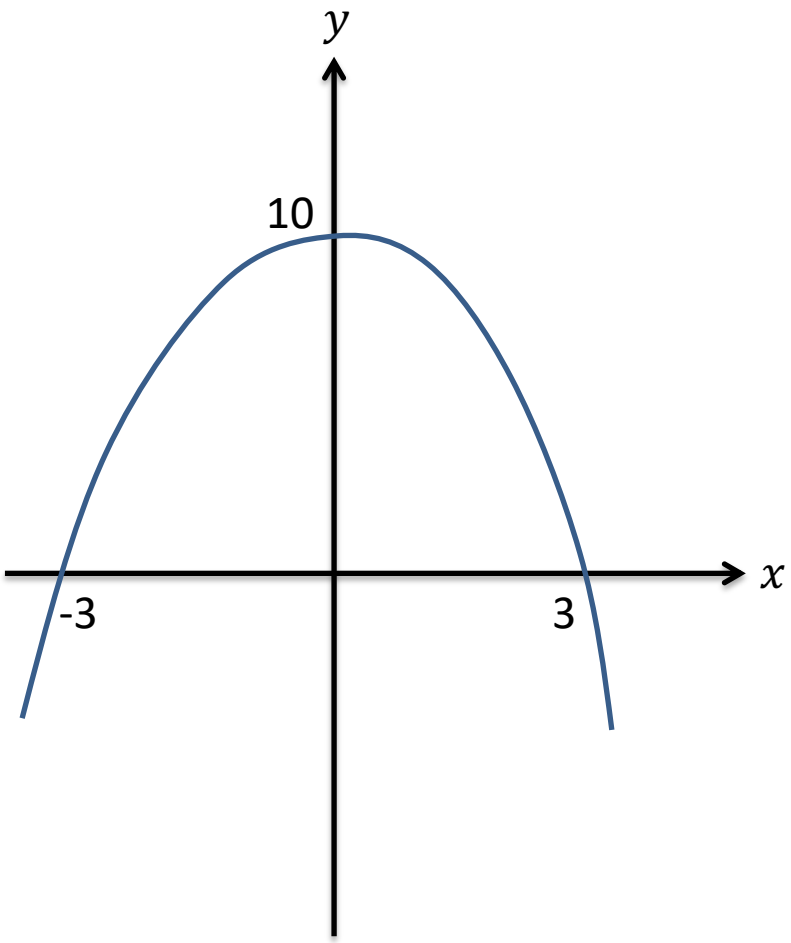
Determine the equation of this quadratic graph, in the form $y = ax^2 + bx + c$.

Further Example



Determine the equation of this quadratic graph, in the form $y = ax^2 + bx + c$, where a, b, c are integers.

Further Example



Determine an equation of this quadratic graph.

Challenge Problems

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Non-Calculator Questions

Into how many regions is the plane divided when the following three parabolas are drawn?

$$y = x^2$$
$$y = x^2 - 2x$$
$$y = x^2 + 2x + 2$$

Example

How many **distinct** real solutions do each of the following have?

$a)\ x^2 - 12x + 36 = 0$

$b)\ x^2 + x + 3 = 0$

$c)\ x^2 - 2x - 1 = 0$

5 : The Discriminant

$$ax^2 + bx + c = 0$$

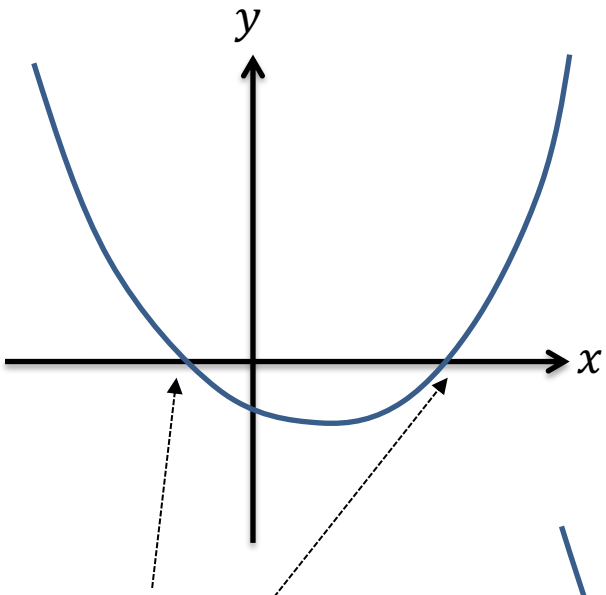
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Looking at this formula, when in general do you think we have:

- No real roots?
- Equal roots?
- Two distinct roots?

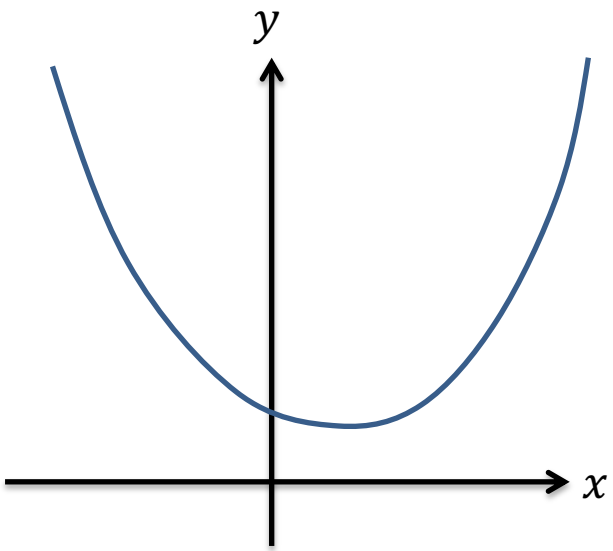
$b^2 - 4ac$ is known as the discriminant.

The Discriminant



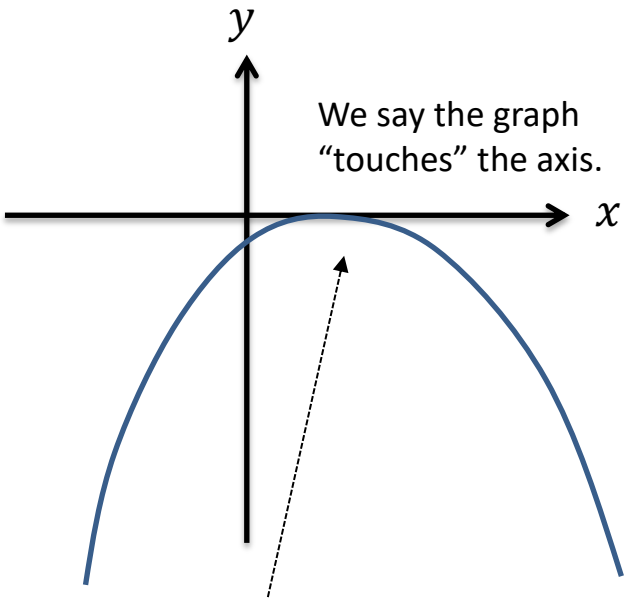
Distinct real roots

$$b^2 - 4ac > 0$$



No real roots

$$b^2 - 4ac < 0$$



Equal roots

$$b^2 - 4ac = 0$$

Example

Equation	Discriminant	Number of Distinct Real Roots
$x^2 + 3x + 4 = 0$		
$x^2 - 4x + 1 = 0$		
$x^2 - 4x + 4 = 0$		
$2x^2 - 6x - 3 = 0$		
$x - 4 - 3x^2 = 0$		
$1 - x^2 = 0$		

Problems involving the discriminant

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The equation $x^2 + 2px + (3p + 4) = 0$, where p is a positive constant, has equal roots.

(*a*) Find the value of p.

(4)

(*b*) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$.

(2)

Further Example

$$x^2 + 5kx + (10k + 5) = 0$$

where k is a constant.

Given that this equation has equal roots, determine the value of k.

Find the range of values of k for which $x^2 + 6x + k = 0$ has two distinct real solutions.

Challenge Problems

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Non-Calculator Questions

1 Given a real constant c, the equation

$$x^4 = (x - c)^2$$

Has four real solutions (including possible repeated roots) for:

A) $c \leq \frac{1}{4}$

B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$

C) $c \leq -\frac{1}{4}$

D) all values of c

2 The equation $(2 + x - x^2)^2 = 16$ has how many real root(s)?

3 A rectangle has perimeter P and area A. The values P and A must satisfy:

A) $P^3 > A$

B) $A^2 > 2P + 1$

C) $P^2 \geq 16A$

D) $PA > A + P$

A spear is thrown over level ground from the top of a tower.
The height, in metres, of the spear above the ground after t seconds is modelled by the function: $h(t) = 12.25 + 14.7t - 4.9t^2, \quad t \geq 0$

a) Interpret the meaning of the constant term 12.25 in the model.
b) After how many seconds does the spear hit the ground?
c) Write $h(t)$ in the form $A - B(t - C)^2$, where A, B and C are constants to be found.
d) Using your answer to part c or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached?